

MULTI-INSTANTON CALCULUS IN SUPERSYMMETRIC THEORIES

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In this talk I review some recent results concerning multi-instanton calculus in supersymmetric field theories. More in detail, I will show how these computations can be efficiently performed using the formalism of topological field theories.

1 Introduction

Our understanding of the non-perturbative sector of field and string theories has greatly progressed in recent times. Recently, for the first time, the entire non-perturbative contribution to the holomorphic part of the Wilsonian effective action was computed for $N = 2$ globally supersymmetric (SUSY) theories with gauge group $SU(2)$, using ansätze dictated by physical intuitions¹. A few years later, a better understanding of non-perturbative configurations in string theory led to the conjecture that certain IIB string theory correlators on an $AdS_5 \times S^5$ background are related to Green's functions of composite operators of an $N = 4$ $SU(N_c)$ Super Yang-Mills (SYM) theory in four dimensions in the large N_c limit². Although supported by many arguments, these remarkable results remain conjectures and a clear mathematical proof seems to be out of reach at the moment. In our opinion this state of affairs is mainly due to the lack of adequate computational tools in the non-perturbative region. To the extent of our knowledge, the only way to perform computations in this regime in SUSY theories and from first principles is via multi-instanton calculus. Using this tool, many partial checks have been performed on these conjectures, both in $N = 2$ and $N = 4$ SUSY gauge theories^{3, 4, 5, 6, 7}. The limits on these computations come from the exploding amount of algebraic manipulations to be performed and from the lack of an explicit parametrization of instantons of

winding number greater than two⁸. In order to develop new computational tools that might allow an extension to arbitrary winding number, I revisit instanton computations for $N = 2$ in the light of the topological theory built out of $N = 2$ SYM⁹, *i.e.* the so-called Topological Yang-Mills theory (TYM)¹⁰.

2 Topological Yang-Mills Theory

Here I collect some results which will be relevant to our discussion. I use the same notation of⁹ to which I refer the reader for a detailed exposition of this material. As it is well known¹¹, after the twisting procedure, the Lagrangian of $N = 2$ SYM is invariant under

$$\begin{aligned} sA &= \psi - Dc, \\ s\psi &= -[c, \psi] - D\phi, \\ s\phi &= -[c, \phi], \\ sc &= -\frac{1}{2}[c, c] + \phi. \end{aligned} \quad (1)$$

The BRST operator, s , defined in (1) is such that $s^2 = 0$ and when the set of equations in (1) is restricted to the solutions of the Euler-Lagrange classical equation (*the zero modes*) it gives the derivative on the space M^+ . In terms of the parameters of the ADHM construction, these solutions look like

$$A = U^\dagger dU, \quad (2)$$

$$\psi = U^\dagger \mathcal{M} f(d\Delta^\dagger)U + U^\dagger (d\Delta) f \mathcal{M}^\dagger U, \quad (3)$$

$$c = U^\dagger (s + C)U, \quad (4)$$

$$\phi = U^\dagger \mathcal{M} f \mathcal{M}^\dagger U + U^\dagger A U. \quad (5)$$

Plugging (5) into (1) leads to the action of the operator s on the elements of the ADHM construction

$$\mathcal{M} = s\Delta + C\Delta = \mathcal{S}\Delta \quad , \quad (6)$$

$$\mathcal{A} = s\mathcal{M}\Delta + C\mathcal{M} = \mathcal{S}\mathcal{M} \quad , \quad (7)$$

$$s\mathcal{A} = -[C, \mathcal{A}] \quad , \quad (8)$$

$$sC = \mathcal{A} - CC \quad , \quad (9)$$

i.e. this is the realization of the BRST algebra on the instanton moduli space. C is a connection I must introduce to have a nilpotent s .

Let us see now how, at the semi-classical level, any correlator which is expressed as a polynomial in the fields, becomes after projection onto the zero-mode subspace, a well-defined differential form on M^+ ¹⁰. Symbolically

$$\langle fields \rangle = \int_{M^+} [(fields) e^{-S_{\text{TYM}}}]_{\text{zero-mode}} \quad (10)$$

A generic function on the zero-mode subspace will then have the expansion

$$g(\widehat{\Delta}, \widehat{\mathcal{M}}) = g_0(\widehat{\Delta}) + g_{i_1}(\widehat{\Delta}) \widehat{\mathcal{M}}_{i_1} + \dots + \frac{1}{p!} g_{i_1 i_2 \dots i_p}(\widehat{\Delta}) \widehat{\mathcal{M}}_{i_1} \widehat{\mathcal{M}}_{i_2} \dots \widehat{\mathcal{M}}_{i_p} \quad (11)$$

the coefficients of the expansion being totally antisymmetric in their indices. Now $\widehat{\mathcal{M}}_i$'s and the $s\widehat{\Delta}_i$'s are related by a (moduli-dependent) linear transformation K_{ij} , which is completely known once the explicit expression for C is plugged into the $\widehat{\mathcal{M}}_i$'s:

$$\widehat{\mathcal{M}}_i = K_{ij}(\widehat{\Delta}) s\widehat{\Delta}_j \quad . \quad (12)$$

It then follows that

$$\begin{aligned} \widehat{\mathcal{M}}_{i_1} \widehat{\mathcal{M}}_{i_2} \dots \widehat{\mathcal{M}}_{i_p} &= \\ &= \epsilon_{j_1 \dots j_p} K_{i_1 j_1} K_{i_2 j_2} \dots K_{i_p j_p} s^p \widehat{\Delta} = \\ &= \epsilon_{i_1 \dots i_p} (\det K) s^p \widehat{\Delta} \quad , \end{aligned} \quad (13)$$

where $s^p \widehat{\Delta} \equiv s\widehat{\Delta}_1 \dots s\widehat{\Delta}_p$. I then conclude that

$$\begin{aligned} \int_{M^+} g(\widehat{\Delta}, \widehat{\mathcal{M}}) &= \frac{1}{p!} \int_{M^+} g_{i_1 \dots i_p}(\widehat{\Delta}) \widehat{\mathcal{M}}_{i_1} \dots \widehat{\mathcal{M}}_{i_p} \\ &= \int_{M^+} s^p \widehat{\Delta} |\det K| g_{12 \dots p}(\widehat{\Delta}) \quad . \end{aligned} \quad (14)$$

The determinant of K naturally stands out as *the instanton integration measure for $N = 2$ SYM theories*.

3 Conclusions

In this talk i have argued that the results of multi-instanton calculus at the semiclassical level can be easily recovered in the formalism of topological field theories. More benefits come from this reformulation than what I have presented until now. It is for example possible to show that correlators of the type of (10) can be written as a total derivative on the moduli space. The only contributions to these quantities, come from zero-size instantons. Given the peculiar properties of the ADHM construction at the boundary of the moduli space, this might lead to recursion relations among correlators computed at different winding number. To get to this conclusion, we probably have to better understand what is the geometrical role of the action, projected on the subspace of the zero-modes. The connection between moduli spaces of instantons and their construction in terms of D-branes of string theory can be very helpful in this respect.

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